MATH 2040A Lecture 3 (Sep 14, 2016)

§ Linear Transformations

Given V, W vector spaces $/ \mathbb{F}$

a map $T: V \longrightarrow W$ is linear transformation

if it "respects" $+$ & $\cdot$, ie

(i) $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y}) \quad \forall \vec{x}, \vec{y} \in V$

(ii) $T(a \cdot \vec{x}) = a \cdot T(\vec{x}) \quad \forall \vec{x} \in V, \forall a \in \mathbb{F}$

Examples (1) Identity $I_V : V \to V \quad I_V(\vec{x}) = \vec{x}$

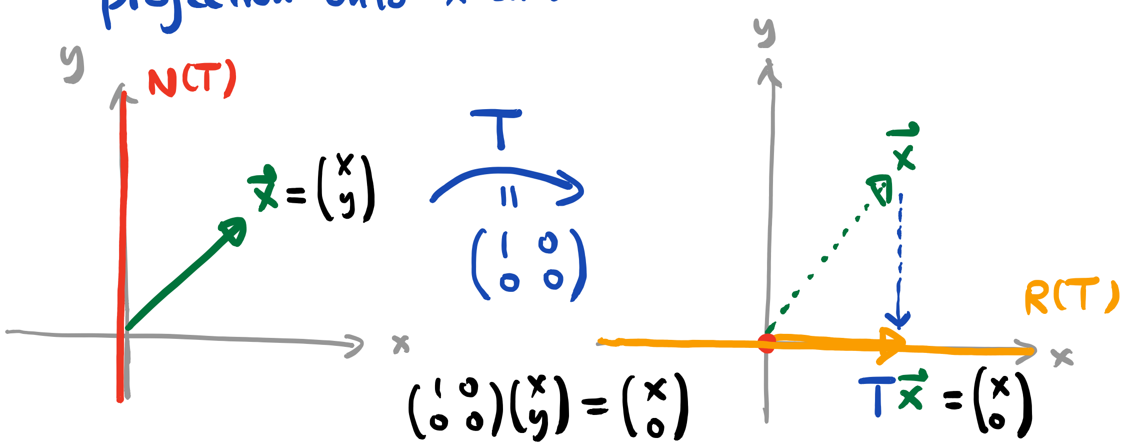
(2) Zero. $T_0 : V \to W \quad T_0(\vec{x}) = \vec{0}_W$

(3) Geometric linear transformations:

e.g. rotations, reflections, dilations etc.

~~translation~~ not linear $\because T(\vec{0}_V) = \vec{0}_W$.

"projection onto x-axis"



Given $T: V \to W$

$\leadsto$ 2 important subspaces:

null space: $N(T) = \{ \vec{x} \in V : T(\vec{x}) = \vec{0}_W \} \subseteq V$
(kernel)

image space: $R(T) = \{ T(\vec{x}) : \vec{x} \in V \} \subseteq W$
(range)

Dim. Theorem: $\text{Nullity}(T) + \text{rank}(T) = \dim V$
(rank-nullity thm)

$$\underbrace{\text{Nullity}(T)}_{= \dim N(T)} + \underbrace{\text{rank}(T)}_{= \dim R(T)} = \dim \underbrace{V}_{\text{domain}}$$

FACTS: (a) T 1-1 $\iff$ $N(T) = \{\vec{0}_V\}$

(b) T onto $\iff$ $R(T) = W$

Def: T isomorphism if T is both 1-1 & onto.

(c) If $T: V \to W$ where $\dim V = \dim W$,

then T 1-1 $\iff$ T onto.

An important example:

Fix $A \in M_{m \times n}(\mathbb{F})$, we can define a map

$L_A = T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ linear.

$\vec{x} \longmapsto A\vec{x}$

Important Theorem: Fix V, W $\dim V = n$, $\dim W = m$.

$$\left\{\begin{array}{c} \text{linear transf.} \\ T : V \to W \end{array}\right\}^{+,\cdot,\circ} \xleftrightarrow{\cong_\beta^\gamma} \left\{\begin{array}{c} A \in M_{m\times n}(\mathbb{F}) \\ m\times n \text{ matrices} \end{array}\right\}^{+,\cdot,AB}$$

$$L_A \xleftarrow{?} A$$

$$T \xmapsto[?]{\beta,\gamma} A$$

§ Matrix representation of T

Recall: V $\quad \beta = \{\vec{v}_1, \dots, \vec{v}_n\}$: basis for V $\quad \dim V = n$

$$\begin{array}{ccc} V & \xrightarrow{\cong_\beta} & \mathbb{F}^n \\ \vec{v} & \longmapsto & [\vec{v}]_\beta = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \end{array}$$

$$\vec{v} = a_1 \vec{v}_1 + \cdots + a_n \vec{v}_n$$

coordinate representation of $\vec{v}$ w.r.t. β

Now, given a linear $T : \underset{\substack{\cup \\ \beta}}{V} \to \underset{\substack{\cup \\ \gamma}}{W}$.

"bases": β , γ

$$
\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, leftrightarrow, "\cong_\beta"'] \arrow[dr, phantom, "\circlearrowright"] & W \arrow[d, leftrightarrow, "\cong_\gamma"] \\
\mathbb{F}^n \arrow[r, "L_A"] & \mathbb{F}^m
\end{tikzcd}
$$

$$\boxed{A = [T]_\beta^\gamma}$$

matrix representation of T w.r.t. bases β & γ.

Computational: $T : V \longrightarrow W$

$\beta = \{\vec{v}_1, \ldots, \vec{v}_n\}$ $\qquad$ $\gamma = \{\vec{w}_1, \ldots, \vec{w}_m\}$

$$[T]_\beta^\gamma = \underbrace{\begin{pmatrix} | & | & & | \\ [T\vec{v}_1]_\gamma & [T\vec{v}_2]_\gamma & \cdots & [T\vec{v}_n]_\gamma \\ | & | & & | \end{pmatrix}}_{\in M_{m \times n}(\mathbb{F})}$$

Interesting Facts:

(1) $[T \pm U]_\beta^\gamma = [T]_\beta^\gamma \pm [U]_\beta^\gamma$

(2) $[a \cdot T]_\beta^\gamma = a \cdot [T]_\beta^\gamma$

(3) $[T \circ U]_\beta^\alpha = [T]_\gamma^\alpha [U]_\beta^\gamma$

$$\begin{tikzcd} V \arrow[r, "U"] \arrow[rr, bend left, "T \circ U"] & W \arrow[r, "T"] & Z \\ \beta & \gamma & \alpha \end{tikzcd}$$

Recall:

$AB \neq BA$

$T \circ U \neq U \circ T$